K-STABILITY AND GIT FOR CUBIC HYPERSURFACES

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Abstract. In this note, we exhibit the proof that K-(semi/poly)stability of cubic hypersurfaces of dimension at least five coincide with the corresponding GIT stability. In particular, their K-moduli and GIT moduli stacks are isomorphic. The main step is to show that every K-semistable $\mathbb{Q}$-Gorenstein limit of smooth cubic hypersurfaces is again a cubic hypersurface. We establish Cartierness of the limiting hyperplane divisor using local log canonical thresholds and mixed intersections. When its linear system has full-dimensional image, the codimension of the base locus strengthens the threshold estimate enough to apply the normalized-volume bound for local complete intersections. In the remaining case, tangent-sheaf semistability and adjunction theory reduce the image to a Segre cone and the geometry of its general fiber then gives the required Cartierness.

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1. Introduction

1.1. **Background and main result.** K-stability, introduced by Tian [Tia97] and reformulated algebraically by Donaldson [Don02], relates the existence of Kähler–Einstein metrics to the geometry of degenerations of Fano varieties. The algebraic theory now provides moduli stacks of K-semistable Fano varieties with projective good moduli spaces parametrizing K-polystable varieties; see [ABHLX20, BLX22, BX19, XZ20, Jia20, LWX21, LXZ22] and the references therein. Describing these moduli spaces explicitly, however, requires control of the varieties that occur on their boundary.

For several families, the moduli continuity method identifies the K-moduli compactification with a classical GIT quotient. Examples include del Pezzo surfaces [OSS16], intersections of two quadrics [SS17], cubic threefolds [LX19], and cubic fourfolds [Liu22]. A central ingredient is the comparison between the anticanonical volume of a K-semistable Fano variety and the normalized volumes of its singularities. The latter invariant was introduced by C. Li [Li18]. For varieties of large anticanonical volume, this comparison places strong restrictions on possible degenerations.

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In this paper we study cubic hypersurfaces in higher dimensions. If $X \subset \mathbb{P}^{n+1}$ is a smooth cubic n-fold and $H = \mathcal{O}_X(1)$, then

$$-K_X \sim (n-1)H, \qquad H^n = 3.$$

Liu–Xu [LX19, Theorem 1.1] and Liu [Liu22, Theorem 1.1] proved that K-stability and GIT stability agree in dimensions three and four, respectively. Kong–Shen–Zhao–Zhao [KSZZ26a] established the corresponding result for cubic fivefolds. We prove the following statement uniformly for all $n \geq 5$.

Let $\mathcal{M}^K_{\mathrm{cubic}}$ denote the K-moduli stack parametrizing K-semistable Fano n-folds of anticanonical volume $3(n-1)^n$ that admit a $\mathbb{Q}$-Gorenstein smoothing to a smooth cubic. Let U^{ss} be the GIT-semistable locus in the projective space of cubic forms on $\mathbb{P}^{n+1}$, and set

$$\mathcal{M}^{\mathrm{GIT}}_{\mathrm{cubic}} = [U^{\mathrm{ss}}/\operatorname{PGL}(n+2)].$$

We test GIT stability using $\mathrm{SL}(n+2)$ and use the effective projective action for the quotient stack.

Theorem 1.1. *Let $n \geq 5$, and let $X = V(f) \subset \mathbb{P}^{n+1}$ be a cubic hypersurface. Then X is K-semistable (respectively, K-polystable or K-stable) if and only if the cubic form $[f]$ is GIT-semistable (respectively, GIT-polystable or GIT-stable) for the natural action of $\mathrm{SL}(n+2)$. Moreover, there is a natural isomorphism $\mathcal{M}^K_{\mathrm{cubic}} \cong \mathcal{M}^{\mathrm{GIT}}_{\mathrm{cubic}}$.*

Remark 1.2. The implication from K-stability to GIT stability is standard for Fano hypersurfaces as the CM polarization is a positive rational multiple of the usual GIT polarization; see [PT06] and the discussion following [Liu22, Theorem 1.1]. The content of Theorem 1.1 is the converse and the identification of the compactifications. In particular, every smooth cubic n-fold is K-stable, since smooth cubics are GIT-stable.

1.2. **Outline of the proof.** The main difficulty is to show that the K-moduli boundary still parametrizes cubic hypersurfaces. Let X be a K-semistable $\mathbb{Q}$-Gorenstein limit of smooth cubic n-folds. By [Liu22, Theorem 3.1], the limiting hyperplane class is an ample integral $\mathbb{Q}$-Cartier Weil divisor L satisfying

$$-K_X \sim (n-1)L, \qquad L^n = 3, \qquad h^0(X, L) = n+2,$$

and $2L$ is Cartier. The classification of del Pezzo varieties then reduces the problem to proving that L itself is Cartier. Our main geometric result is the following.

Theorem 1.3. *Let X be a K-semistable $\mathbb{Q}$-Gorenstein limit of smooth cubic n-folds, where $n \geq 5$. Then its limiting hyperplane divisor L is Cartier, and $|L|$ embeds X as a cubic hypersurface in $\mathbb{P}^{n+1}$.*

Proof. Let L be the divisor supplied by Theorem 3.1. Then $-K_X \sim (n-1)L$, $L^n = 3$, $h^0(X, L) = n+2$, and $2L$ is Cartier. Put $W = \overline{\phi_{|L|}(X)}$. Then we have $\dim W = n$ by Corollary 6.13 and the cartierness of L follows from Proposition 5.5. The embedding follows from [Fuj90, Corollary 1.7]. □

Proof of Theorem 1.1. This follows from the moduli continuity argument of [LX19, proof of Theorem 1.1] and also [Liu22, proof of Theorem 1.1]. □

The local obstruction is particularly useful. At a non-Cartier point, the index 1 cover associated to L is a nonsmooth Gorenstein klt germ whose normalized volume is strictly greater than $2(n-1)^n$. It therefore cannot be a local complete intersection, by [Liu22, Theorem 1.3]. On the other hand, we show that a Gorenstein klt germ of dimension c with a proper ideal of

log canonical threshold greater than $c-2$ is a hypersurface germ (Lemma 4.5). The proof cuts down to a Gorenstein klt surface and uses the classification of Du Val singularities. Thus a sufficiently strong threshold bound for the lifted sections of L forces Cartierness.

Write $\mathcal{B}$ for the geometric base locus of $|L|$. To obtain the threshold bound, set

$$W = \overline{\phi_{|L|}(X)} \subset \mathbb{P}^{n+1}.$$

On the normalized graph $p : Y \to X$, write $P = p^*L = M + F$, where M and F are the free and fixed parts of the pulled-back degree-one system, and put $a_i = P^{n-i}M^i$. The mixed-intersection estimate gives the following lower bound for the threshold of the degree-one base ideal on each index 1 cover:

$$\theta = \frac{n-1}{3(n+1)} \sum_{i=0}^{n} a_i.$$

The useful additional observation is that if $\mathcal{B}$ has codimension k, then $a_0 = \cdots = a_{k-1} = 3$, while the local threshold bound gives $\theta \le k$. These two statements can be used successively to improve both the codimension and the threshold.

Suppose first that $\dim W = n$. Nondegeneracy gives $a_n \ge 2$. Together with the preceding equalities, this first forces $k \ge n-1$ and then gives $\theta > n-2$. The local criterion now applies at every non-Cartier point, proving Cartierness of L in this case. This is the argument of Section 5.

When $m = \dim W < n$, the mixed intersections vanish for $i > m$, so a different input is needed. Semistability of T_X shows that $K_{B'} + (m-1)A'$ is not pseudoeffective on a smooth model of the Stein base. Adjunction theory, followed by slope inequalities for the two rulings of a scroll, leaves only degree-three cones over $\mathbb{P}^1 \times \mathbb{P}^2$ in dimensions $n \in \{5, 6, 7\}$. The general resolved fiber C is rational, and $P \cdot C$ is either $1/2$ or 1. If $P \cdot C = 1/2$, the threshold estimate contradicts the discrepancy computed by adjunction along C. If $P \cdot C = 1$, monotonicity forces $a_0 = \cdots = a_{n-1} = 3$. Relative ampleness on the normalized graph then makes the base locus finite, and the local criterion again proves Cartierness. We carry this out in Section 6.

1.3. **Relation to earlier work.** The volume bound underlying the preceding discussion is the following conjecture of Spotti–Sun [SS17, Conjecture 5.5]; see also [LX19, Conjecture 4.5].

Conjecture 1.4 (ODP gap conjecture)**.** Let $x \in Z$ be a nonsmooth n-dimensional klt singularity, with $n \ge 2$. Then

$$\widehat{\mathrm{vol}}(x, Z) \le 2(n-1)^n,$$

with equality if and only if $x \in Z$ is an ordinary double point.

Together with the finite degree formula for normalized volumes, this conjecture would give the required Cartierness directly. Liu–Xu [LX19] proved the three-dimensional case using the classification of canonical and terminal singularities. Liu [Liu22] proved the local complete intersection case in arbitrary dimension and treated cubic fourfolds by studying the limiting linear systems through non-vanishing and degree-six K3 surfaces. Our local criterion follows the approach suggested in [Liu22, Remark 4.8], where higher codimensional alpha invariants and surface sections are used to control the index 1 covers.

Our global threshold estimate comes from the fivefold argument of Kong–Shen–Zhao–Zhao [KSZZ26a]. The contribution here is to relate this estimate systematically to the codimension of the base locus, and to use foliations and adjunction theory for images of smaller dimension. These arguments do not require K_X to be Cartier. They use the established volume bound for local complete intersections, rather than the general ODP gap conjecture.

1.4. **An independent proof.** Kong–Shen–Zhao–Zhao [KSZZ26b, Theorem 1.1] give an independent proof of the K-moduli–GIT comparison for cubic hypersurfaces in arbitrary dimension. Both proofs use Liu's reduction to Cartierness of the limiting hyperplane divisor, the normalized-volume bound for local complete intersections, and tangent-sheaf semistability.

For $2 \leq \dim W < n$, they combine slope semistability with vanishing and Koszul resolutions to prove that W has minimal degree, then use the classification of such varieties, alpha bounds, and estimates for S-invariants to exclude the possible images [KSZZ26b, Propositions 3.1 and 3.4]. Our argument uses kernel foliations and adjunction theory, followed by slope inequalities for the two rulings and the geometry of the general fiber.

For $\dim W = n$, their proof separates cubic and quadric images and studies the quadrics according to their rank [KSZZ26b, Theorem 4.1]. For quadrics of rank at least four, threshold and multiplicity bounds, together with Abhyankar's inequality, force the index-one cover to be a local complete intersection. We derive a uniform threshold bound from the codimension of the base locus and obtain the local criterion by cutting to a Gorenstein klt surface.

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AI disclosure. We are sharing this manuscript with everyone interested in the problem, in the spirit of sharing mathematical knowledge.

We do not claim any traditional academic credit for this paper.

This paper is not intended for publication in a traditional mathematics journal.

LLM(AI) contributed substantive mathematical suggestions, including proof strategies and arguments used in preparing the manuscript. Their principal contribution concerned the lower-dimensional image case as they suggested combining the foliations defined by the rational maps with tangent-sheaf semistability and adjunction theory.

The author reviewed the resulting arguments, removed unnecessary detail, and revised the exposition. In particular, the analysis in Section 5 and Section 6 was simplified. The author takes full responsibility for the mathematical content and its presentation, including any remaining errors or omissions.

This work owes a fundamental intellectual debt to Yuchen Liu and Chenyang Xu. Liu's study of cubic degenerations through the limiting hyperplane divisor and its linear system is the starting point of our argument. Xu's work with his collaborators on algebraic K-stability and K-moduli supplies the framework in which it is carried out. The use of AI tools does not alter the attribution of these ideas or the results of other researchers used below.

2. Preliminaries

We work over $\mathbb{C}$. Unless otherwise specified, all varieties are assumed to be normal and projective. However, we do not assume normality of the image of a rational map. We always write X for a $\mathbb{Q}$-Gorenstein limit of smooth cubic hypersurfaces.

2.1. **Divisors, linear systems, and local thresholds.** For an integral Weil divisor D on Z, we denote $\mathcal{O}_Z(D)$ by its rank-one reflexive sheaf, and denote $H^i(Z, \mathcal{O}_Z(D))$ by $H^i(Z, D)$. We set $h^i(Z, D) = \dim_{\mathbb{C}} H^i(Z, D)$. For a coherent sheaf $\mathcal{E}$, we write $\mathcal{E}^* = \mathcal{H}om(\mathcal{E}, \mathcal{O}_Z)$ for its dual and $\mathcal{E}^{**}$ for its double dual. The relations $\sim$, $\sim_{\mathbb{Q}}$, and $\equiv$ are referred to linear, rational linear, and numerical equivalence, respectively. For divisor classes, an inequality $D_1 \geq D_2$ means

that $D_1 - D_2$ has an effective representative. A real Cartier divisor is pseudoeffective if its numerical class lies in the closure of the cone generated by effective divisors.

We use the quotient convention for projective bundles: if $\varpi : \mathbb{P}_T(\mathcal{E}) \to T$ is the projective bundle of a vector bundle $\mathcal{E}$, then $\mathcal{O}_{\mathbb{P}_T(\mathcal{E})}(1)$ is its tautological quotient line bundle and $\varpi_* \mathcal{O}_{\mathbb{P}_T(\mathcal{E})}(1) = \mathcal{E}$.

Definition 2.1 (Weil linear systems and their base loci)**.** A nonzero subspace $V \subset H^0(Z, D)$ determines a linear system $\mathcal{V} \subset |D|$ of effective Weil divisors. Its geometric base locus is denoted by

$$\mathcal{B}(\mathcal{V}) = \bigcap_{D' \in \mathcal{V}} \operatorname{Supp} D'.$$

If D is Cartier, its base ideal is

$$\mathfrak{b}(\mathcal{V}) = \operatorname{im}\big(V \otimes \mathcal{O}_Z(-D) \longrightarrow \mathcal{O}_Z\big).$$

For a Weil divisor, $\mathcal{B}(|D|)$ always means the geometric base locus. It contains the non-Cartier locus of D as a member avoiding a point gives a local trivialization of $\mathcal{O}_Z(D)$ there. Evaluation away from $\mathcal{B}(\mathcal{V})$ defines the rational map $\phi_{\mathcal{V}} : Z \dashrightarrow \mathbb{P}(V)$, with the quotient convention above. For the complete system we write $\phi_{|D|}$.

Let Z be a normal variety. Let D be an effective $\mathbb{Q}$-divisor on Z, such that $K_Z + \Delta$ is $\mathbb{Q}$-Cartier. Then (X, D) is called a log pair. We say that E is a prime divisor over Z if E is a divisor on a normal variety Y where $f : Y \longrightarrow Z$ is a proper birational morphism. We define the log discrepancy of E with rrespect to (Z, D) as

$$A_{(Z,D)}(E) = 1 + \operatorname{ord}_E\big(K_Y - \pi^*(K_Z + D)\big).$$

We also write $c_Z(E) = \pi(E)$ be the center of E with respect to (Z, D).

For an ideal $\mathfrak{a}$, the order $\operatorname{ord}_E(\mathfrak{a})$ is the minimum of the vanishing orders of its local generators along the center of E. For a local ring R, we write $\operatorname{ht}_R(I)$ for the height of an ideal I and $\operatorname{edim}(R) = \dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$ for its embedding dimension, where $\mathfrak{m}$ is the maximal ideal. A subideal $Q \subset I$ is a reduction of I if $I^{j+1} = QI^j$ for some $j \geq 0$; equivalently, Q and I have the same integral closure.

Definition 2.2 ($\mathbb{Q}$-Fano varieties)**.** A $\mathbb{Q}$-*Fano variety* is a normal projective klt variety Z with ample $\mathbb{Q}$-Cartier anticanonical divisor $-K_Z$. A *log Fano pair* (Z, Δ) consists of a projective klt pair with effective $\mathbb{Q}$-divisor Δ and ample $-(K_Z + \Delta)$.

Definition 2.3 (Log canonical thresholds and multiplier ideals)**.** Let Z be a $\mathbb{Q}$-Gorenstein klt variety. For a nonzero coherent ideal $\mathfrak{a} \subset \mathcal{O}_Z$, set

$$\operatorname{lct}(Z; \mathfrak{a}) := \inf_{\operatorname{ord}_E(\mathfrak{a}) > 0} \frac{A_Z(E)}{\operatorname{ord}_E(\mathfrak{a})}.$$

The log canonical threshold $\operatorname{lct}_x(Z; \mathfrak{a})$ is defined by divisors whose centers contain x. For an effective $\mathbb{Q}$-Cartier divisor D, or a linear system $\mathcal{V}$ of such divisors, we replace by $\operatorname{ord}_E(D)$, or $\operatorname{ord}_E(\mathcal{V}) = \min_{D' \in \mathcal{V}} \operatorname{ord}_E(D')$ in the denominator. When $\mathcal{V}$ is a subsystem of a Cartier linear system, its threshold is that of $\mathfrak{b}(\mathcal{V})$. It satisfied the following coefficients scale properties:

$$\operatorname{lct}(Z; \lambda\mathcal{V}) = \lambda^{-1} \operatorname{lct}(Z; \mathcal{V}) \qquad (\lambda > 0).$$

More generally, for a log pair (Z, Δ) and a log resolution $\pi : Y \to Z$ with $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$, one can define the multiplier ideal as following:

$$\mathcal{J}\big((Z, \Delta); \mathfrak{a}^\lambda\big) = \pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_Z + \Delta) - \lambda F \rceil).$$

We omit Δ when it is zero; products of ideal powers are defined by subtracting the corresponding sum of divisors on a common log resolution.

2.2. **K-stability and K-moduli.**

Definition 2.4 (Valuative invariants)**.** Let Z be an n-dimensional $\mathbb{Q}$-Fano variety. Let L be an ample $\mathbb{Q}$-Cartier divisor, and assume E be a prime divisor on a smooth projective birational model $\pi : Y \to Z$. Define the S-invariant:

$$S_L(E) = \frac{1}{L^n} \int_0^\infty \mathrm{vol}\big(\pi^* L - tE\big)\, dt.$$

Changing variables in the integral gives $S_{cL}(E) = cS_L(E)$ for $c \in \mathbb{Q}_{>0}$. We write $S_Z(E) = S_{-K_Z}(E)$ and define the K-stability threshold by

$$\delta(Z) = \inf_E \frac{A_Z(E)}{S_Z(E)},$$

where E runs over all prime divisors over Z. For a log Fano pair (Z, Δ), one defines $\delta(Z, \Delta)$ by using $A_{Z,\Delta}(E)$ and $S_{-(K_Z+\Delta)}(E)$.

Theorem 2.5 ([Fuj19, Li17])**.** *A $\mathbb{Q}$-Fano variety Z is K-semistable if and only if $\delta(Z) \geq 1$, equivalently if and only if*

$$A_Z(E) \geq S_{-K_Z}(E)$$

for every prime divisor E over Z.

Definition 2.6 (Alpha invariants)**.** For an ample $\mathbb{Q}$-Cartier divisor L on an n-dimensional projective klt variety Z, we define

$$\alpha(Z; L) = \inf\{\mathrm{lct}(Z; D) \mid D \geq 0,\ D \sim_{\mathbb{Q}} L\},$$

where D ranges over effective $\mathbb{Q}$-divisors. If Z is $\mathbb{Q}$-Fano, we write $\alpha(Z) = \alpha(Z; -K_Z)$. For $1 \leq k \leq n$, the higher codimensional alpha invariant is defined by

$$\alpha^{(k)}(Z) = \inf_{r,\mathcal{V}} \mathrm{lct}\Big(Z; \frac{1}{r}\mathcal{V}\Big),$$

where $r > 0$ is an integer with $-rK_Z$ Cartier and $\mathcal{V} \subset |-rK_Z|$ has base scheme of codimension at least k. Equivalently, one takes the infimum over codimension-k complete intersections of members of $|-rK_Z|$; see [Zhu18, Definition 3.1 and Remark 3.1]. In particular, $\alpha^{(1)}(Z) = \alpha(Z)$, and

$$\mathrm{lct}\Big(Z; \frac{1}{r}\mathcal{V}\Big) = r\ \mathrm{lct}\big(Z; \mathfrak{b}(\mathcal{V})\big), \qquad \alpha(Z; cL) = c^{-1}\alpha(Z; L) \quad (c \in \mathbb{Q}_{>0}).$$

Theorem 2.7 ([Zhu18, Theorem A])**.** *If Z is an n-dimensional K-semistable $\mathbb{Q}$-Fano variety, then*

$$\alpha^{(k)}(Z) \geq \frac{k}{n+1} \qquad (1 \leq k \leq n).$$

In particular, if $-K_Z \sim_{\mathbb{Q}} (n-1)L$, then $\alpha(Z; L) \geq (n-1)/(n+1)$.

The comparison with GIT uses three facts from K-moduli theory: existence of polystable replacements, properness of good moduli spaces, and openness of K-semistability. Furthermore, people prove the following K-moduli theorem:

Theorem 2.8. [LWX21] *Fix $n \in \mathbb{N}_{>0}$ and $v \in \mathbb{Q}_{>0}$. The moduli functor which sends any scheme S to*

$$\left\{ \mathcal{X}/S \;\middle|\; \begin{array}{c} \mathcal{X} \to S \text{ is a proper flat family satisfied Kollár condition,} \\ \text{each fiber } \mathcal{X}_s \text{ is } n\text{-dimensional K-semistable } \mathbb{Q}\text{-Fano variety} \\ \text{of volume } v \end{array} \right\}$$

can be represented by an Artin stack of finite type over $\mathbb{C}$. Denoted by $\mathcal{M}^K_{n,v}$. Moreover, it admits a projective good moduli space $M^K_{n,v}$ parameterizing all K-polystable $\mathbb{Q}$-Fano varieties.

Remark 2.9 (Smoothable families). In the $\mathbb{Q}$-Gorenstein smoothable setting, proper good moduli spaces were constructed in [LWX19, Theorem 1.3]. For smoothable log pairs with boundary proportional to the anticanonical divisor, existence of K-semistable fillings is established in [ADL19, Theorem 3.19]. Uniqueness of K-polystable limits in families is proved in [BX19].

2.3. **Singularities and normalized volumes.**

Definition 2.10 (Normalized volumes)**.** Let $x \in Z$ be an n-dimensional klt germ and let E be a prime divisor over Z centered at x. Its valuation ideals and volume are defined by:

$$\mathfrak{a}_j(E) = \{f \in \mathcal{O}_{Z,x} \mid \operatorname{ord}_E(f) \geq j\}, \qquad \operatorname{vol}_{Z,x}(E) = \limsup_{j\to\infty} \frac{\ell\big(\mathcal{O}_{Z,x}/\mathfrak{a}_j(E)\big)}{j^n/n!},$$

where ℓ denotes length. Define

$$\widehat{\operatorname{vol}}_{Z,x}(E) = A_Z(E)^n \operatorname{vol}_{Z,x}(E), \qquad \widehat{\operatorname{vol}}(x, Z) = \inf_{c_Z(E)=x} \widehat{\operatorname{vol}}_{Z,x}(E).$$

The latter is called the local volume of germ.

Theorem 2.11. *The following normalized-volume estimates hold.*

(1) [Liu22, Theorem 2.14]. *If Z is an n-dimensional K-semistable $\mathbb{Q}$-Fano variety, then every closed point $x \in Z$ satisfies*

$$(-K_Z)^n \leq \left(1 + \frac{1}{n}\right)^n \widehat{\operatorname{vol}}(x, Z).$$

(2) [Liu22, Theorem 2.13]. *If $\tau : (y \in Y) \to (x \in Z)$ is a finite quasi-étale Galois morphism between klt singularities, then $\widehat{\operatorname{vol}}(y, Y) = \deg(\tau)\widehat{\operatorname{vol}}(x, Z)$.*

(3) [Liu22, Theorem 2.12]. *If $y \in Y$ is a non-smooth local complete intersection klt singularity of dimension n, then*

$$\widehat{\operatorname{vol}}(y, Y) \leq 2(n-1)^n,$$

with equality if and only if it is an ordinary double point.

2.4. **Polarized models and foliations.**

Definition 2.12 (Polarized birational models)**.** A quasi-polarized pair (B, A) consists of a normal projective variety and a nef and big Cartier divisor. Two such pairs are said to be polarized birationally equivalent if their polarizations have the same pullback to a common resolution; see [And13, Definition 3.1]. This identifies their section spaces and images of complete linear-series maps.

For a terminal $\mathbb{Q}$-factorial pair of dimension $m \geq 2$, a *zero reduction* is the last pair in a $(K_B + \Delta)$-minimal model program with scaling, where (B, Δ) is klt and $\Delta \sim_{\mathbb{Q}} (m-1)A$. The polarization descends through each birational step, so a zero reduction is polarized birationally equivalent to (B, A); see [And13, Proposition 4.2, Definition 4.4, and Remark 4.5(i)].

Definition 2.13 (Slopes)**.** Let L be an ample $\mathbb{Q}$-Cartier divisor on an n-dimensional normal projective variety Z. The slope with respect to L of a torsion-free sheaf $\mathcal{E}$ is defined by

$$\mu_L(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot L^{n-1}}{\mathrm{rk}(\mathcal{E})}.$$

Here $c_1(\mathcal{E})$ is the Weil divisor class of the reflexive determinant $\det \mathcal{E} := (\bigwedge^{\mathrm{rk}(\mathcal{E})} \mathcal{E})^{**}$; its intersection with L^{n-1} is defined even if that class is not $\mathbb{Q}$-Cartier. The sheaf is said to be slope semistable if $\mu_L(\mathcal{F}) \leq \mu_L(\mathcal{E})$ for every coherent subsheaf with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$. Scaling L by a positive rational number does not change semistability.

Definition 2.14 (Foliations)**.** Write Ω^1_Z for the sheaf of Kähler differentials and $T_Z = (\Omega^1_Z)^*$ for the reflexive tangent sheaf. A subsheaf is *saturated* if $T_Z = (\Omega^1_Z)^*$ quotient by this subsheaf is torsion-free; its saturation is the smallest saturated subsheaf containing it.

A *foliation* is a saturated subsheaf $\mathcal{F} \subset T_Z$ which is closed under the Lie bracket. The *kernel foliation* of a dominant rational map is the saturation of the subsheaf tangent to its general fibers. Its leaves are the maximal connected integral submanifolds on the locus where both Z and the foliation are regular.

3. Limits of cubic hypersurfaces

3.1. **The structure theorem for cubic limits.** We start with Liu's structure theorem. It supplies the limiting hyperplane class and reduces the possible failure of Cartierness to local index two.

Theorem 3.1 ([Liu22, Theorem 3.1])**.** *Let $n \geq 4$ and let $\mathcal{X} \to B$ be a K-semistable $\mathbb{Q}$-Fano family over a smooth pointed curve $0 \in B$, whose restriction over $B \setminus \{0\}$ is a smooth family of cubic n-folds. Put $X = \mathcal{X}_0$. There is a $\mathbb{Q}$-Cartier integral Weil divisor class L on X with the following properties.*

1. *$\mathcal{O}_X(mL)$ is Cohen–Macaulay for every integer m.*
2. *$-K_X \sim (n-1)L$ and $L^n = 3$.*
3. *$h^i(X, \mathcal{O}_X(mL)) = h^i(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(m))$ for every integer m, every $i \geq 0$ and every $b \neq 0$. In particular $h^0(X, L) = n + 2$ and the intermediate cohomology vanishes for all m.*
4. *Every $\mathbb{Q}$-Cartier Weil divisor D on X has $2D$ Cartier.*

Definition 3.2 (The cubic limit pair)**.** Throughout the paper, (X, L) denotes a K-semistable $\mathbb{Q}$-Gorenstein limit of smooth cubic n-folds with $n \geq 5$, together with the ample integral $\mathbb{Q}$-Cartier divisor L supplied by Theorem 3.1, unless other hypotheses are stated. In particular, the non-Cartier locus of L is contained in the geometric base locus $\mathcal{B}(|L|)$.

3.2. **The non-Cartier index 1 covers.** The next lemma explains why it suffices to make the local covers complete intersections. The volume comparison is immediate once we know that the cover is singular; excluding a smooth cover is the step that uses the smoothing.

Lemma 3.3. *In the setting of Theorem 3.1, suppose L is not Cartier at $x \in X$, and let $\tau : (y \in Y) \to (x \in X)$ be its index 1 cover. Then $\deg(\tau) = 2$, $y \in Y$ is non-smooth, and*

$$\widehat{\mathrm{vol}}(y, Y) \geq 6\Big(\frac{n(n-1)}{n+1}\Big)^n > 2(n-1)^n.$$

Thus any points on the cover cannot be local complete intersection singularities.

Proof. By Theorem 3.1(4), the local Cartier index of L at x is two. Thus τ is a quasi-étale double cover with a unique point y over x. Since $K_X \sim -(n-1)L$ and $\tau^{[*]}L$ is Cartier, we have

$$K_Y = \tau^* K_X \sim -(n-1)\tau^{[*]}L,$$

so K_Y is Cartier at y. Since τ is quasi-étale and X is klt, Y is also klt and hence Cohen–Macaulay. Together with the Cartierness of K_Y, this shows that Y is Gorenstein at y.

We next show that Y is not smooth at y. Suppose, for a contradiction, that Y is smooth at y. After a finite base change if necessary, the construction in the proof of [Liu22, Theorem 3.1] provides a Weil divisor $\mathcal{L}$ on $\mathcal{X}$ extending L, such that $\mathcal{O}_{\mathcal{X}}(j\mathcal{L})$ is Cohen–Macaulay and restricts to $\mathcal{O}_X(jL)$ for every $j \in \mathbb{Z}$. In particular,

$$\mathcal{O}_{\mathcal{X}}(2\mathcal{L})/t\mathcal{O}_{\mathcal{X}}(2\mathcal{L}) \cong \mathcal{O}_X(2L),$$

where t is a local parameter of B at 0. The right-hand side is invertible, so Nakayama's lemma shows that the rank-one torsion-free module $\mathcal{O}_{\mathcal{X}}(2\mathcal{L})_x$ is cyclic, hence free over the local domain $\mathcal{O}_{\mathcal{X},x}$. Thus $2\mathcal{L}$ is Cartier near x.

After shrinking around x, we can choose a trivialization of $\mathcal{O}_{\mathcal{X}}(2\mathcal{L})$ compatible with the one defining τ. The associated cyclic algebra

$$\mathcal{A} = \mathcal{O}_{\mathcal{X}} \oplus \mathcal{O}_{\mathcal{X}}(\mathcal{L})$$

defines a finite double cover $\mathcal{Y} \to \mathcal{X}$. Since both summands of $\mathcal{A}$ are flat over B, the morphism $\mathcal{Y} \to B$ is flat. The restriction isomorphisms identify its central fiber with the original index-one cover $\tau : Y \to X$. Over $t \neq 0$, the cover is étale because $\mathcal{L}$ is Cartier there.

Then $\mathcal{Y}$ is smooth at y and $(dt)_y \neq 0$. The deck involution ι fixes y and satisfies $\iota^* t = t$. Choose an eigenbasis of the cotangent space $\mathfrak{m}_y/\mathfrak{m}_y^2$ containing the class of t. Averaging lifts of these eigenvectors, we obtain a regular system of parameters

$$(t, u_1, \dots, u_a, v_1, \dots, v_b)$$

at y such that

$$\iota^* t = t, \qquad \iota^* u_i = u_i, \qquad \iota^* v_j = -v_j.$$

Therefore, near y, the fixed locus $F = \mathrm{Fix}(\iota)$ is cut out by $v_1 = \cdots = v_b = 0$. In particular, F is smooth at y, and $t|_F$ is part of a regular system of parameters at y. Hence F is not contained in the central fiber and meets $\{t \neq 0\}$. This contradicts the fact that ι acts freely on the étale double cover over $\{t \neq 0\}$.

Now we can apply the local-to-global inequality and the finite degree formula in Theorem 2.11(1)–(2). Since $(-K_X)^n = 3(n-1)^n$, this give the asserted lower bound. Its ratio to $2(n-1)^n$ is $3\big(n/(n+1)\big)^n > 1$. By Statement (3) we can excludes a local complete intersection point on the cover.

□

4. Cartierness from local thresholds

We prove two forms of the local Cartierness criterion. Higher codimensional alpha invariants will give Cartierness at the generic points of the largest base components (Proposition 4.8). A stronger, pointwise threshold bound will exclude every non-Cartier point (Proposition 4.9). Both arguments use Lemma 4.5, which reduces a Gorenstein klt germ to a surface by general Cartier sections.

4.1. **Reduction of singularity & Local algebra.** We first recall the local algebra needed for this reduction.

Theorem 4.1 ([LLS08, Corollary 1.4(i)]). *Let (Z,Δ) be a normal log pair. Let $\mathfrak{c},\mathfrak{b}$ be ideals. If $f_1,\dots,f_s\in\mathfrak{c}$ generate a reduction of $\mathfrak{c}$, then for every integer $m\geq s$ and every real $\lambda>0$,*

$$\mathcal{J}\big((Z,\Delta);\mathfrak{c}^m\mathfrak{b}^\lambda\big)=(f_1,\dots,f_s)\,\mathcal{J}\big((Z,\Delta);\mathfrak{c}^{m-1}\mathfrak{b}^\lambda\big).$$

The case without the auxiliary ideal is obtained by taking $\mathfrak{b}=\mathcal{O}_Z$.

Theorem 4.2 ([NR54]). *If $(R,\mathfrak{m})$ is a Noetherian local ring of dimension h with infinite residue field and J is $\mathfrak{m}$-primary, then J has a reduction generated by h elements.*

Theorem 4.3 ([Kol13, Section 3.3]). *A two-dimensional Gorenstein klt singularity over a characteristic-zero field is geometrically smooth or Du Val. Its embedding dimension is at most three, and its completed local ring is regular or a hypersurface.*

Lemma 4.4. *Let Z be a normal $\mathbb{Q}$-Gorenstein klt variety over a field of characteristic zero, let $\mathfrak{a}\subsetneq\mathcal{O}_Z$ be a coherent ideal, and let C be an irreducible component of $V(\mathfrak{a})$ of codimension h. At the generic point η_C one has*

$$\mathrm{lct}_{\eta_C}(Z;\mathfrak{a})\leq h.$$

In particular, if every local log canonical threshold of $\mathfrak{a}$ is greater than a real number q, then every irreducible component of $V(\mathfrak{a})$ has codimension greater than q.

Proof. Set $R=\mathcal{O}_{Z,\eta_C}$. This is an h-dimensional normal $\mathbb{Q}$-Gorenstein klt local ring, and $\mathfrak{a}R$ is primary for its maximal ideal. Its residue field is an extension of the characteristic-zero ground field and is therefore infinite. The minimal-reduction theorem (Theorem 4.2) gives $f_1,\dots,f_h\in\mathfrak{a}R$ which generate a reduction Q of $\mathfrak{a}R$.

Apply the singular Skoda equality of Theorem 4.1 with $\Delta=0$, $s=m=h$, and no auxiliary ideal. It gives

$$\mathcal{J}\big(Z;(\mathfrak{a}R)^h\big)=Q\,\mathcal{J}\big(Z;(\mathfrak{a}R)^{h-1}\big)\subseteq Q\subsetneq R.$$

Thus the pair with ideal coefficient h is not klt. Since the ambient germ is klt, the klt threshold and the log canonical threshold agree as a supremum, so $\mathrm{lct}_{\eta_C}(Z;\mathfrak{a})\leq h$. The final assertion follows by applying this inequality at the generic point of every component. □

Lemma 4.5. *Let (Z,z) be a Gorenstein klt local germ of dimension $c\geq 2$ over a characteristic-zero field, and let $I\subset\mathcal{O}_{Z,z}$ be a proper ideal. Assume*

$$\mathrm{lct}_z(Z;I)>c-2.$$

Then Z is a local complete intersection at z. More precisely, its completed local ring is regular or a hypersurface.

Proof. If $c=2$, the assertion is Theorem 4.3. Suppose $c\geq 3$ and put $r=c-2$. Choose a rational number $0<\epsilon<1$ such that

$$r+\epsilon<\mathrm{lct}_z(Z;I).$$

By Lemma 4.4, $\mathrm{ht}(I)>r$. Since Z is Cohen–Macaulay, r sufficiently general linear combinations $f_1,\dots,f_r$ of generators of I form a regular sequence. Set $Z_j=V(f_1,\dots,f_j)$ and $I_j=I\mathcal{O}_{Z_j}$.

We claim that each Z_j is Gorenstein klt and that $(Z_j,I_j^{r-j+\epsilon})$ is klt near z. This holds for $j=0$ by the choice of ϵ. For the induction step, the height inequality

$$\mathrm{ht}(I_j)=\mathrm{ht}(I)-j>r-j\geq 1$$

ensures that I_j has no divisorial component. Let $H_{j+1} = V(f_{j+1}|_{Z_j})$ be a general Cartier member. On a log resolution of I_j, it has the same vanishing orders as I_j along the exceptional divisors, and its free part is smooth by Bertini. Replacing one factor of I_j by H_{j+1} therefore preserves the strict discrepancy inequalities and gives a plt pair

$$\big(Z_j, H_{j+1} + I_j^{r-j-1+\epsilon}\big)$$

near z. Here the notation records an ideal with its indicated exponent. Cartier adjunction then gives the asserted klt pair on $Z_{j+1} = H_{j+1}$; see [Kol97, Theorems 4.8 and 7.5(1)]. Gorensteinness is preserved by each Cartier cut.

Thus $S = Z_r$ is a Gorenstein klt surface germ. The strict inequality in the hypothesis is used to retain klt singularities after all r cuts. By Theorem 4.3, $\operatorname{edim}(\mathcal{O}_{S,z}) \le 3$. Quotienting a local ring by r elements lowers its embedding dimension by at most r, so

$$\operatorname{edim}(\mathcal{O}_{Z,z}) - c \le \operatorname{edim}(\mathcal{O}_{S,z}) - 2 \le 1.$$

Since $\mathcal{O}_{Z,z}$ is Cohen–Macaulay, a minimal Cohen presentation of its completion has kernel generated by at most one element. Hence the completion is regular or a hypersurface, and Z is a local complete intersection at z. □

4.2. **Local thresholds and Cartierness.** The preceding lemma applies in the local dimension, which may be smaller than n at a generic point of the base locus. This is why the higher codimensional alpha bound is sufficient for the next two statements.

Let L be an integral $\mathbb{Q}$-Cartier Weil divisor on a normal projective klt variety Z, and put $V = H^0(Z, L) \neq 0$. If L has local Cartier index r at $z \in Z$, denote by

$$\tau : (\widetilde{z} \in \widetilde{Z}) \longrightarrow (z \in Z)$$

the index 1 cover associated to L, of degree r. All such covers are understood as covers of germs. Put $\widetilde{L} = \tau^{[*]}L$, which is Cartier.

Definition 4.6. The pullbacks of the sections in the fixed space V define the coherent ideal sheaf

$$\mathfrak{I} = \operatorname{im}\big(V \otimes \mathcal{O}_{\widetilde{Z}}(-\widetilde{L}) \longrightarrow \mathcal{O}_{\widetilde{Z}}\big), \qquad I_{\widetilde{z}} \coloneqq \mathfrak{I}_{\widetilde{z}} \subset \mathcal{O}_{\widetilde{Z},\widetilde{z}}.$$

We call $I_{\widetilde{z}}$ the *local degree-one base ideal*.

After trivializing $\widetilde{L}$ near $\widetilde{z}$, it is generated by the functions representing the pullbacks of all global sections of L. Where L is Cartier, we take $r = 1$ and the identity cover.

Let w be a normalized divisorial valuation over $\widetilde{Z}$ whose center contains $\widetilde{z}$. Write $w|_{\mathbb{C}(Z)} = e\,\operatorname{ord}_G$, where G is a prime divisor over Z and $e = e(w/G)$ is the ramification index. Since the cover is cyclic of degree r, we have $e \mid r$. Quasi-étaleness and crepancy give

$$A_{\widetilde{Z}}(w) = eA_Z(G),$$
$$w(I_{\widetilde{z}}) = e\,\operatorname{ord}_G(|L|),$$

where $\operatorname{ord}_G(|L|) = \min_{D\in|L|} \operatorname{ord}_G(D)$. Thus the ratio of log discrepancy to vanishing order is preserved, and

$$\operatorname{lct}_{\widetilde{z}}(\widetilde{Z}; I_{\widetilde{z}}) = \operatorname{lct}_z(Z; |L|).$$

Lemma 4.7. *Let Z be an n-dimensional K-semistable $\mathbb{Q}$-Fano variety, and let L be an integral $\mathbb{Q}$-Cartier Weil divisor such that*

$$-K_Z \sim (n-1)L, \qquad 2L \text{ is Cartier.}$$

Assume $|L| \neq \varnothing$ and $\mathcal{B} \neq \varnothing$, and put $k = \operatorname{codim}_Z \mathcal{B}$. Fix a non-Cartier point $z \in Z$, so that L has local index two at z, and let $\tau : (\widetilde{z} \in \widetilde{Z}) \to (z \in Z)$ be its local index 1 cover. Let $I_{\widetilde{z}} \subset \mathcal{O}_{\widetilde{Z},\widetilde{z}}$ be the local degree-one base ideal. Then

$$\operatorname{lct}_{\widetilde{z}}(\widetilde{Z}; I_{\widetilde{z}}) \geq \frac{k(n-1)}{n+1} > k-2.$$

Proof. Choose a positive integer ℓ divisible by both 2 and $n-1$, and set $q = \ell/(n-1)$. The image

$$V_\ell = \operatorname{im}\Big(\operatorname{Sym}^\ell H^0\big(Z, \mathcal{O}_Z(L)\big) \longrightarrow H^0\big(Z, \mathcal{O}_Z(\ell L)\big)\Big)$$

is a linear subsystem of $|\ell L| = |-qK_Z|$. Its set-theoretic base locus is $\mathcal{B}$ as a degree-one section nonvanishing at a point gives a nonvanishing ℓ-th power, while at a degree-one base point every product vanishes. Set $\mathfrak{b}_\ell = \mathfrak{b}(V_\ell)$, a coherent ideal sheaf on Z. The base scheme of V_ℓ has codimension at least k. Applying Zhu's theorem (Theorem 2.7) on higher codimensional alpha invariants and using the scaling rule for log canonical thresholds, we obtain

$$q\, \operatorname{lct}\big(Z; \mathfrak{b}_\ell\big) = \operatorname{lct}\Big(Z, \frac{1}{q} V_\ell\Big) \geq \frac{k}{n+1}.$$

Pullback is compatible with multiplication under the identification $\tau^*\mathcal{O}_Z(\ell L) \simeq \mathcal{O}_{\widetilde{Z}}(\ell \widetilde{L})$. Thus, writing $\mathfrak{I}$ for the ideal sheaf on the chosen cover, we have

$$\mathfrak{b}_\ell \mathcal{O}_{\widetilde{Z}} = \mathfrak{I}^\ell, \qquad \mathfrak{b}_\ell \mathcal{O}_{\widetilde{Z},\widetilde{z}} = I_{\widetilde{z}}^\ell.$$

Since the quasi-étale crepant cover preserves thresholds,

$$\operatorname{lct}_z\big(Z; \mathfrak{b}_\ell\big) = \operatorname{lct}_{\widetilde{z}}(\widetilde{Z}; I_{\widetilde{z}}^\ell) = \frac{1}{\ell} \operatorname{lct}_{\widetilde{z}}(\widetilde{Z}; I_{\widetilde{z}}).$$

Since the local threshold is at least the global threshold, and $\ell = q(n-1)$, it follows that

$$\operatorname{lct}_{\widetilde{z}}(\widetilde{Z}; I_{\widetilde{z}}) \geq \frac{\ell k}{q(n+1)} = \frac{k(n-1)}{n+1}.$$

Finally,

$$\frac{k(n-1)}{n+1} - (k-2) = \frac{2(n+1-k)}{n+1} > 0,$$

because $k \leq n$. □

Proposition 4.8. *Let (X, L) be a K-semistable $\mathbb{Q}$-Gorenstein limit of smooth cubic n-folds, $n \geq 4$, with L as in Theorem 3.1. If $\mathcal{B}$ is nonempty, set $k = \operatorname{codim}_X \mathcal{B}$. Then every irreducible component of $\mathcal{B}$ of codimension k is generically contained in the Cartier locus of L.*

In particular, if $\mathcal{B}$ is finite, then L is Cartier.

Proof. Assume first that $\mathcal{B}$ is nonempty, and let C be a codimension-k component with generic point η. If $k = 1$, the Cartierness at η can be deduced from the normality of X. In the following let's assume that $k \geq 2$ and L is not Cartier at η.

Take the index 1 cover $\tau_\eta : (\widetilde{\eta} \in \widetilde{X}) \to (\eta \in X)$, which has degree two. The local ring $\mathcal{O}_{\widetilde{X},\widetilde{\eta}}$ has dimension k and is Gorenstein and klt. The ideal $I_{\widetilde{\eta}}$ of Definition 4.6, formed from $H^0(X, L)$, is primary for its maximal ideal. By Lemma 4.7 we have:

$$\operatorname{lct}_{\widetilde{\eta}}(\widetilde{X}; I_{\widetilde{\eta}}) > k-2.$$

Thereore, applying Lemma 4.5, in dimension k, we deduce that the cover is local complete intersection at $\widetilde{\eta}$.

Trivializing $2L$ near η extends the index cover to a finite cover $\tau : \widetilde{U} \to U$. By openness of the local complete intersection locus and finiteness of τ, we may shrink U so that $\widetilde{U}$ is a local complete intersection. The non-Cartier locus of L is closed and contains η, hence C. Thus any closed point of $C \cap U$ contradicts Lemma 3.3. This proves the first assertion.

If $\mathcal{B}$ is finite, the first assertion shows that L is Cartier at every point of $\mathcal{B}$. □

Proposition 4.9. *Let (X, L) be as in Definition 3.2, and let $x \in X$ be a closed non-Cartier point of L. On its index 1 cover $\tau : (\widetilde{x} \in \widetilde{X}) \to (x \in X)$, the local degree-one base ideal $I_{\widetilde{x}}$ satisfies*

$$\operatorname{lct}_{\widetilde{x}}(\widetilde{X}; I_{\widetilde{x}}) \le n - 2.$$

Proof. The cover is Gorenstein klt, and $I_{\widetilde{x}}$ is proper because a non-Cartier point belongs to $\mathcal{B}$. If its threshold were greater than $n-2$, Lemma 4.5 would make the cover a local complete intersection, contrary to Lemma 3.3. □

5. Linear systems and mixed intersections

The normalized graph of a linear system separates its moving and fixed parts. We use their mixed intersections to estimate the codimension of the base locus and, under K-semistability, the log canonical threshold of the ideal of the lifted sections on index 1 covers. We then apply these estimates to linear systems with full-dimensional image and prove the Cartierness of the hyperplane class on K-semistable cubic limits.

Throughout this section, let Z be a normal projective variety of dimension $n \ge 2$, and let L be an ample integral $\mathbb{Q}$-Cartier Weil divisor with $H^0(Z, L) \ne 0$.

5.1. The normalized graph.

Set

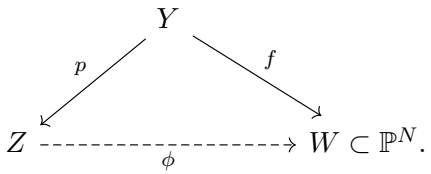
$$V = H^0(Z, L), \qquad N = \dim V - 1, \qquad \phi = \phi_{|L|}, \qquad W = \overline{\phi(Z)} \subset \mathbb{P}^N,$$

and write $m = \dim W$. As above, $\mathcal{B}$ denotes the geometric base locus of $|L|$. Choose a positive integer ℓ such that ℓL is Cartier. All base loci are geometric base loci in the sense of Definition 2.1. Since we use the complete linear system, W is integral and nondegenerate; we make no assumption on its normality.

Let

$$V_\ell = \operatorname{im}\big(\operatorname{Sym}^\ell V \longrightarrow H^0(Z, \ell L)\big), \qquad \mathfrak{b}_\ell = \operatorname{im}\big(V_\ell \otimes \mathcal{O}_Z(-\ell L) \longrightarrow \mathcal{O}_Z\big).$$

Take Y to be the normalization of the blowup of $\mathfrak{b}_\ell$, thus Y is the normalized graph, independent of ℓ, with morphisms $p : Y \to Z$ and $f : Y \to W$ fitting into the diagram



Let $A_W = \mathcal{O}_W(1)$. Put $P = p^*L$ and $M = f^*A_W$. Both P and M are nef, and M is Cartier, globally generated, and p-ample. Moreover,

$$P = M + F, \qquad F \ge 0, \qquad \mathfrak{b}_\ell \mathcal{O}_Y = \mathcal{O}_Y(-\ell F) \text{ Cartier.}$$

The support satisfies $\operatorname{Supp} F = p^{-1}(\mathcal{B})$ set-theoretically, and F is the fixed $\mathbb{Q}$-divisor of P.

5.2. **Mixed intersections and threshold estimates.** Define the mixed intersections on the normalized graph Y by

$$a_i = P^{n-i}M^i \qquad (0 \le i \le n).$$

They are unchanged by pullback to any higher birational model. On such a model, we continue to write P, M, F for their pullbacks; when comparing two models, we write the pullbacks explicitly. For a prime divisor G over Z, let b_G be the minimal vanishing order along G of a member of $|L|$. On a model carrying G, this is the coefficient of G in the pull-back of the fixed divisor F.

We first show how the mixed intersections reflect the dimension of the image and the codimension of the base locus.

Lemma 5.1. *The mixed intersections satisfy*

$$L^n = a_0 \ge a_1 \ge \cdots \ge a_n \ge 0, \qquad a_i = 0 \quad \text{if } i > m.$$

If $\mathcal{B} \neq \varnothing$ and $k = \operatorname{codim}_Z \mathcal{B}$, then

$$a_0 = \cdots = a_{k-1} = L^n > a_k.$$

Proof. Since $F = P - M$ is effective and P, M are nef,

$$a_i - a_{i+1} = F \cdot P^{n-i-1}M^i \ge 0 \qquad (0 \le i \le n-1).$$

Also $a_0 = L^n$, and $a_i = 0$ for $i > m$ because M is pulled back from W.

Suppose $\mathcal{B}$ has codimension k. For $0 \le i \le k-2$, the effective cycle $F \cdot M^i$ has dimension $n-1-i > n-k$, whereas its image under p lies in $\mathcal{B}$. Hence $p_*(F \cdot M^i) = 0$. The projection formula gives

$$a_i - a_{i+1} = L^{n-i-1} \cdot p_*(F \cdot M^i) = 0,$$

proving the equalities. For $k = 1$ there is only $a_0 = L^n$.

To prove the strict inequality, choose an irreducible component T of $\mathcal{B}$ of codimension k, and let η_T be its generic point. Since $\operatorname{Supp} F = p^{-1}(\mathcal{B})$, some prime component D of F dominates T. Its generic fiber D_{η_T} has dimension $k-1$. The projection formula gives

$$D \cdot P^{n-k}M^{k-1} = (L^{n-k} \cdot T)\, \deg\big((M|_{D_{\eta_T}})^{k-1}\big) > 0$$

by the ampleness of L and the p-ampleness of M. All components of F have nonnegative intersection with the nef classes $P^{n-k}M^{k-1}$, and the coefficient of D in F is positive. Hence

$$a_{k-1} - a_k = F \cdot P^{n-k}M^{k-1} > 0.$$

This is the positivity argument of [HP24, proof of Theorem 3.2, Step 2]. □

Thus the codimension of a nonempty base locus is the index of the first strict drop in the mixed-intersection sequence. In particular, equality of the first n mixed intersections characterizes finiteness of the base locus.

Corollary 5.2. *The geometric base locus $\mathcal{B}$ is finite if and only if*

$$a_0 = \cdots = a_{n-1} = L^n.$$

Proof. If $\mathcal{B} = \varnothing$, then $F = 0$, so $P = M$ and all the mixed intersections equal L^n. Otherwise, Lemma 5.1 shows that the displayed equalities hold if and only if $\operatorname{codim}_Z \mathcal{B} = n$, which is equivalent to finiteness. □

The segment joining P and M gives a lower bound for the volume integral defining S_L in terms of the mixed intersections. This is the key estimate for relating the codimension of the base locus to the log canonical threshold of the lifted base ideal on a index 1 cover.

Lemma 5.3. *For every prime divisor G over Z,*

$$S_L(G) \geq \frac{b_G}{(n+1)L^n} \sum_{i=0}^{n} a_i.$$

Assume in addition that Z is a K-semistable $\mathbb{Q}$-Fano variety and $-K_Z \sim_{\mathbb{Q}} \lambda L$ for some $\lambda \in \mathbb{Q}_{>0}$. Set

$$\theta = \frac{\lambda}{(n+1)L^n} \sum_{i=0}^{n} a_i.$$

Then

$$A_Z(G) \geq S_{-K_Z}(G) \geq \theta \cdot b_G.$$

Proof. Choose a smooth model $\sigma : Y' \to Y$ carrying G, and use P, M, F for the pulled-back divisors. Then $F - b_G G \geq 0$. The volume bound is immediate if $b_G = 0$. Otherwise, for $0 \leq t \leq 1$,

$$P - t \cdot b_G G = (1-t)P + tM + t(F - b_G G) \geq (1-t)P + tM.$$

The divisor on the right is nef. Monotonicity of volume and integration over $0 \leq t b_G \leq b_G$ give

$$\begin{aligned} S_L(G) &\geq \frac{b_G}{L^n} \int_0^1 \big((1-t)P + tM\big)^n \, dt \\ &= \frac{b_G}{(n+1)L^n} \sum_{i=0}^{n} a_i. \end{aligned}$$

Here $\int_0^1 \binom{n}{i}(1-t)^{n-i}t^i \, dt = 1/(n+1)$. Under the additional K-semistable Fano hypotheses, $S_{-K_Z}(G) = \lambda S_L(G)$, and the valuative criterion proves the discrepancy bound. □

On an index 1 cover, the same estimate bounds the log canonical threshold of the lifted base ideal. Comparing it with the local upper bound at the generic point of a base component relates θ to the codimension of the base locus.

Lemma 5.4. *Assume that Z is a K-semistable $\mathbb{Q}$-Fano variety and $-K_Z \sim_{\mathbb{Q}} \lambda L$ for some $\lambda \in \mathbb{Q}_{>0}$, and let θ be as in Lemma 5.3. Fix a non-Cartier point $z \in Z$ of L, of local index r, and let $\tau : (\widetilde{z} \in \widetilde{Z}) \to (z \in Z)$ be its local index 1 cover. The local degree-one base ideal $I_{\widetilde{z}} \subset \mathcal{O}_{\widetilde{Z},\widetilde{z}}$ satisfies*

$$\mathrm{lct}_{\widetilde{z}}(\widetilde{Z}; I_{\widetilde{z}}) \geq \theta.$$

The same bound holds at a Cartier point of L, using $r = 1$ and the identity cover. If $\mathcal{B} \neq \varnothing$ and $k = \mathrm{codim}_Z \mathcal{B}$, then $\theta \leq k$.

Proof. As in the proof of Lemma 4.7, we have

$$\mathfrak{b}_\ell \mathcal{O}_{\widetilde{Z},\widetilde{z}} = I_{\widetilde{z}}^{\ell},$$

where $\mathfrak{b}_\ell$ is the base ideal of Section 5.1. By Lemma 5.3, we obtain $A_{\widetilde{Z}}(w) \geq \theta w(I_{\widetilde{z}})$. The valuative formula for the log canonical threshold gives the asserted local bound, including the identity-cover case.

Now let η be the generic point of a codimension-k component of $\mathcal{B}$. Choose a index 1 cover $\tau_\eta : (\widetilde{\eta} \in \widetilde{Z}_\eta) \to (\eta \in Z)$, and form $I_{\widetilde{\eta}}$ from the same space $H^0(Z, L)$. Since the zero set of $\mathfrak{b}_\ell$ is $\mathcal{B}$ and

$$\mathfrak{b}_\ell \mathcal{O}_{\widetilde{Z}_\eta,\widetilde{\eta}} = I_{\widetilde{\eta}}^{\ell},$$

the ideal $I_{\widetilde{\eta}}$ is primary for the maximal ideal of the k-dimensional klt local ring $\mathcal{O}_{\widetilde{Z}_\eta,\widetilde{\eta}}$. Apply the local bound just proved at η. Thus Lemma 4.4 gives

$$\theta \le \mathrm{lct}_{\widetilde{\eta}}(\widetilde{Z}_\eta; I_{\widetilde{\eta}}) \le k.$$

□

5.3. **The full-dimensional image.** For a full-dimensional image, its degree gives a lower bound for the last mixed intersection. Combining this with Lemmas 5.1 and 5.4 first bounds the codimension of the base locus, and then improves the threshold estimate.

Proposition 5.5. *Let Z be an n-dimensional K-semistable $\mathbb{Q}$-Fano variety, where $n \ge 5$, and let L be an ample integral $\mathbb{Q}$-Cartier Weil divisor such that*

$$-K_Z \sim_{\mathbb{Q}} (n-1)L, \qquad L^n = 3, \qquad h^0(Z,L) = n+2.$$

Assume $\dim W = n$. *Then*

$$\theta = \frac{n-1}{3(n+1)} \sum_{i=0}^{n} a_i > n-2.$$

If $\mathcal{B} \neq \varnothing$, it has codimension at least $n-1$.

In particular, if $(Z,L) = (X,L)$ is a cubic limit pair as in Definition 3.2, then L is Cartier.

Proof. If $\mathcal{B} = \varnothing$, then $F = 0$ and $a_i = 3$ for all i, so $\theta = n-1 > n-2$. We may therefore assume $\mathcal{B} \neq \varnothing$ for the numerical assertions. Put $k = \mathrm{codim}_Z \mathcal{B}$. We first bound k, then substitute that bound back into the estimate for θ.

The nondegenerate image $W \subset \mathbb{P}^{n+1}$ is a hypersurface of degree at least two. If d is the degree of $f : Y \to W$, then

$$a_n = M^n = d \cdot \deg W \ge 2.$$

By Lemma 5.1, every a_i is at least two, while the first k terms equal three. Therefore by Lemma 5.4,

$$k \ge \theta \ge \frac{n-1}{3(n+1)}\big(3k + 2(n+1-k)\big),$$

which yields

$$k \ge \frac{n^2-1}{n+2} = n-2+\frac{3}{n+2} > n-2.$$

Thus $k \ge n-1$. The first $n-1$ mixed intersections now equal three, and the last two are at least two. Substituting gives

$$\theta \ge \frac{n-1}{3(n+1)}\big(3(n-1)+4\big) = n-2+\frac{n+5}{3(n+1)} > n-2.$$

Finally, suppose $(Z,L) = (X,L)$ is a cubic limit pair. The threshold bound on index 1 covers from Lemma 5.4 contradicts Proposition 4.9 at any closed non-Cartier point. Hence L is Cartier. □

6. The lower-dimensional image

In this section, we determine the image W of the rational map $\phi_{|L|} : X \dashrightarrow \mathbb{P}^{n+1}$ when $m < n$. After passing to the Stein base, tangent-sheaf semistability gives an adjunction obstruction. The section count and the alpha bound then reduce the stein base to a scroll. The numerical contrain by its two rulings exclude the cases for $n \ge 8$ and leave only degree-three Segre cones in dimensions five, six, and seven, and their general resolved fibers are rational curves.

These arguments do not require $L^n = 3$ or Cartierness of $2L$. We impose those assumptions in Section 6.3, where the estimates of Section 5.2 force the base locus to be finite.

Throughout this section, unless otherwise stated, let Z be an n-dimensional K-semistable $\mathbb{Q}$-Fano variety with $n \geq 5$, and let L be an ample integral $\mathbb{Q}$-Cartier Weil divisor such that

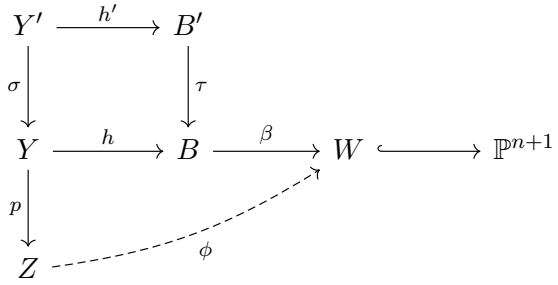
$$-K_Z \sim_{\mathbb{Q}} (n-1)L, \qquad h^0(Z, L) = n + 2.$$

We retain the notation of Section 5.1; in particular, $W \subset \mathbb{P}^{n+1}$ and $m = \dim W$. By Theorem 2.7,

$$\alpha(Z; L) \geq \frac{n-1}{n+1} > \frac{1}{2}.$$

Every cubic limit pair (X, L) satisfies these assumptions by Theorem 3.1.

6.1. **The Stein factorization and adjunction.** Let $p : Y \to Z$ and $f : Y \to W$ be the normalized graph of Section 5.1. Factor f as $Y \xrightarrow{h} B \xrightarrow{\beta} W$, where h has connected fibers and β is finite. The Stein base B is normal. Put $A = \beta^* A_W$, an ample globally generated Cartier divisor, so that $M = h^* A$ on Y. Choose a smooth resolution $\tau : B' \to B$, and resolve the component of $Y \times_B B'$ dominating B' to obtain a smooth Y'. We have a commutative diagram



The morphism h' and τ have connected fibers, since their Stein factors over the normal target is finite birational. Set

$$A' = \tau^* A, \qquad p' = p \circ \sigma, \qquad f' = f \circ \sigma = \beta \circ \tau \circ h'.$$

The divisor A' is nef and big. Using the pullback convention of Section 5.1, on Y' we have

$$P = (p')^* L, \qquad M = (h')^* A' = (f')^* A_W, \qquad F = P - M.$$

On the normalized graph, write $\overline{M} = p_* M$ for the movable Weil divisor class on Z. The projection formula gives

$$\overline{M} = (p')_* (\sigma^* M), \qquad L - \overline{M} = p_* F = (p')_* (\sigma^* F) \geq 0.$$

Completeness of $|L|$ gives

$$H^0(Z, L) \cong H^0(Y, M) \cong H^0(B, A) \cong H^0(B', A') \cong H^0(Y', \sigma^* M).$$

The first identification is part of Section 5.1; the others follow from connected fibers and normality. Thus $\beta : B \to W$ is the map defined by the complete system $|A|$. We will use this to prove that β has degree one.

These constructions use only the general hypotheses at the beginning of Section 5. The K-semistable Fano assumptions enter through the following semistability theorem. Since $-K_Z \sim_{\mathbb{Q}} (n-1)L$, slope semistability with respect to $-K_Z$ is equivalent to slope semistability with respect to L.

Theorem 6.1 ([DGP20, Theorem 2.6(i) and Remark 2.4])**.** *Let Z be a complex K-semistable $\mathbb{Q}$-Fano variety. Then its reflexive tangent sheaf T_Z is slope semistable with respect to $-K_Z$.*

We will show that $K_{B'}+(m-1)A'$ is not pseudoeffective. The following adjunction theorem then restricts the polarized birational type of the Stein base.

Theorem 6.2 ([And13, Theorem 5.1(3) and Proposition 3.5(A)])**.** *Let (B,A) be a quasi-polarized variety of dimension $m\geq 2$ with terminal $\mathbb{Q}$-factorial singularities, where A is a nef and big Cartier divisor. Then $K_B+(m-1)A$ is not pseudoeffective if and only if any zero reduction of (B,A) is one of $\big(\mathbb{P}^m,\mathcal{O}(1)\big)$, a quadric with its hyperplane polarization, a generalized cone over $\big(\mathbb{P}^2,\mathcal{O}(2)\big)$, or a $\mathbb{P}^{m-1}$-bundle over a smooth curve with polarization restricting to $\mathcal{O}(1)$ on the fibers. Here zero reductions and polarized birational equivalence are as in Definition 2.12.*

6.2. **The Stein base and its fibers.** We first establish the adjunction obstruction. This step only uses K-semistability and the relation $-K_Z\sim_{\mathbb{Q}}(n-1)L$; the dimension bound and section count will enter the classification.

Lemma 6.3. *Let Z be an n-dimensional K-semistable $\mathbb{Q}$-Fano variety with $n\geq 2$, and let L be an ample integral $\mathbb{Q}$-Cartier Weil divisor with $H^0(Z,L)\neq 0$. Assume $-K_Z\sim_{\mathbb{Q}}(n-1)L$ and $0<m<n$. For the resolved Stein base (B',A') of Section 6.1, the divisor $K_{B'}+(m-1)A'$ is not pseudoeffective.*

Proof. Use $p':Y'\to Z$ and $h':Y'\to B'$ from Section 6.1. Following [HP24, Lemma 2.6 and Section 6], saturate the image of $(h')^*\Omega^1_{B'}$ in $\Omega^1_{Y'}$. Its determinant is $(h')^*K_{B'}+R$, where $R\geq 0$ as saturation adds the effective divisor contributed by the codimension-one torsion in the quotient. The kernel foliation has rank $n-m$ and first Chern class $-K_{Y'}+(h')^*K_{B'}+R$. Push it to Z and saturate in T_Z. Since $(p')_*K_{Y'}=K_Z$ and saturation can only increase the determinant, the resulting foliation $\mathcal{F}$ satisfies

$$c_1(\mathcal{F})\geq -K_Z+(p')_*(h')^*K_{B'}.$$

If $K_{B'}+tA'$ is pseudoeffective for a nonnegative real number t, its pullback to Y' has nonnegative intersection with P^{n-1}. The projection formula, the preceding inequality, and $\overline{M}\cdot L^{n-1}\leq L^n$ give

$$c_1(\mathcal{F})\cdot L^{n-1}\geq (n-1)L^n-t\overline{M}\cdot L^{n-1}\geq (n-1-t)L^n.$$

By Theorem 6.1, the tangent sheaf is semistable with respect to $-K_Z$, and hence with respect to L. Hence

$$\frac{n-1-t}{n-m}\leq\frac{n-1}{n},\qquad t\geq\frac{m(n-1)}{n}>m-1.$$

The strict inequality is exactly $m(n-1)/n-(m-1)=(n-m)/n>0$. Hence $t=m-1$ is impossible. This also covers $m=1$, in which case it says that $K_{B'}$ is not pseudoeffective. □

Under the standing assumptions of this section, we can now classify the Stein base.

Proposition 6.4. *Assume $m<n$. Let $B\to W$ be the finite map in the Stein factorization of a resolution of $\phi_{|L|}$. Then*

$$m\geq 2,\qquad d=n-m+2\leq m,$$

and the projective image W of $|L|$ is normal and isomorphic to the cone over the Segre embedding of $\mathbb{P}^1\times\mathbb{P}^{d-1}$, with vertex $\mathbb{P}^{m-d-1}$, where a vertex of dimension -1 means no vertex.

The finite map $B\to W$ in the Stein factorization has degree one and is an isomorphism. Equivalently, the polarized Stein base is the ample model of

$$S=\mathbb{P}_{\mathbb{P}^1}\big(\mathcal{O}(1)^{\oplus d}\oplus\mathcal{O}^{\oplus(m-d)}\big),\qquad A_S=\mathcal{O}_S(1).$$

Proof. Since $\alpha(Z;L) > 1/2$, no member $D \in |L|$ contains a prime divisor E on Z with coefficient as least two. Otherwise $A_Z(E) = 1$ would give $\alpha(Z;L) \leq \mathrm{lct}(Z;D) \leq 1/2$.

Lemma 6.5. $\dim W = m \geq 2$.

Proof. If $m = 1$, then $h' \colon Y' \to B'$ is a surjective morphism to a smooth projective curve. Rationality of klt singularities identifies $H^1(Y', \mathcal{O}_{Y'})$ with $H^1(Z, \mathcal{O}_Z)$, and the latter vanishes by Kawamata–Viehweg vanishing. Hodge symmetry therefore gives $H^0(Y', \Omega^1_{Y'}) = 0$.

The pullback map

$$h'^* : H^0(B', \Omega^1_{B'}) \longrightarrow H^0(Y', \Omega^1_{Y'})$$

is injective by the generic smoothnss. Thus $H^0(B', \Omega^1_{B'}) = 0$, so B' has genus zero and $B' \simeq B \simeq \mathbb{P}^1$ as τ has connected fibers. Then $h^0(B', A') = h^0(Z, L) = n + 2$ gives $A' = \mathcal{O}_{\mathbb{P}^1}(n+1)$. A general point of B' lies in the image of the open set where $Z \dashrightarrow W$ is a morphism and p' is an isomorphism, so the pullback of a section of A' vanishes twice at that point, therefore gives a member of $|L|$ with a multiple prime component, contradicting the alpha bound. The existence of such sections is guaranteed by $H^0(B', A'(-2b)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-1)) \neq 0$. Also $m = 0$ is impossible because the sections would all be proportional.

□

For $m \geq 2$, let (B', A') be as in Lemma 6.3, then Theorem 6.2 applies. The projective space and quadric cases have respectively $m+1$ and $m+2$ independent sections, both smaller than $n+2$, and hence are excluded as polarized birational equivalence preserves every section space $H^0(kA')$ and the complete linear-series maps.

The generalized Veronese-cone case is excluded by double hyperplanes. A square of a linear form on $\mathbb{P}^2$ gives a hyperplane section equal to twice the cone over its line. Vary the line and we get a moving family of hyperplane sections whose general member meets the open set where the rational map from Z to the polarized model is a morphism. So its inverse image of the support of the hyperplane section has a codimension-one component on Z and the pulled-back hyperplane has coefficient at least two along that component. This again contradicts the alpha bound.

It remains to consider the projective-bundle alternative. Its base curve is $\mathbb{P}^1$ by the preceding vanishing argument. Up to tensoring a line bundle from the base, we may write $S = \mathbb{P}_{\mathbb{P}^1}\big(\bigoplus_{i=1}^m \mathcal{O}(e_i)\big)$ with $A_S = \mathcal{O}_S(1)$. The polarization is nef, so $e_i \geq 0$. If some $e_i \geq 2$, choose a section supported in that summand with a double zero at a general point of $\mathbb{P}^1$. Its hyperplane divisor contains twice the fiber over that point. The same open-set pullback argument contradicts $\alpha(Z;L) > 1/2$. Therefore all e_i belong to $\{0, 1\}$.

Let $d = \sum_{i=1}^m e_i$. The section count gives

$$n + 2 = h^0(S, A_S) = m + d, \qquad d = n - m + 2, \qquad d \leq m.$$

The complete tautological map of this scroll is birational onto the cone over $\mathbb{P}^1 \times \mathbb{P}^{d-1}$ with the stated vertex. The Segre embedding is projectively normal, so its homogeneous coordinate ring R is integrally closed. The homogeneous coordinate ring of W is the polynomial extension $R[z_1, \ldots, z_{m-d}]$, which is also integrally closed. Hence W is normal.

It remains to identify the Stein base with this image. Polarized birational equivalence identifies all section spaces of B' and S, and therefore identifies their complete linear-series maps. The latter map is birational on S, so the complete map of B also has degree one. Thus $\beta : B \to W$ is finite birational. Normality of the Segre cone now implies that β is an isomorphism. □

The classification alone still permits scrolls of larger degree. Their two rulings give competing lower bounds on divisor classes whose sum is bounded by L. This is the second use of tangent-sheaf semistability.

Write $\varpi : S \to \mathbb{P}^1$ for the bundle projection and $\rho : S \to W$ for the morphism defined by $|A_S|$.

Lemma 6.6. *Assume $m < n$. Use the scroll S and the integer d of Proposition 6.4. There are effective movable Weil divisor classes R_1 and R_2 on Z such that $L - R_1 - R_2$ is effective and*

$$\frac{R_1 \cdot L^{n-1}}{L^n} \geq \frac{n-1}{2n}, \qquad \frac{R_2 \cdot L^{n-1}}{L^n} \geq \frac{(n-1)(d-1)}{nd}.$$

In particular,

$$1 \geq \frac{n-1}{n}\Big(\frac{3}{2} - \frac{1}{d}\Big)$$

or equivalently $d \leq 2 + \frac{4}{n-3}$.

Proof. On S put $R_{1,S} = \varpi^*\mathcal{O}_{\mathbb{P}^1}(1)$ and $R_{2,S} = A_S - R_{1,S}$. The first class gives the map to $\mathbb{P}^1$, while the d sections of $R_{2,S}$ coming from the positive summands give the dominant rational map to $\mathbb{P}^{d-1}$. Resolve both rational maps on a common model dominating Z and S. Take their movable hyperplane divisors and push them to Z, calling the resulting classes R_1 and R_2. The multiplication of a section of $R_{1,S}$ and a section of $R_{2,S}$ is a section of A_S. After resolving and removing fixed parts, this gives $R_1 + R_2 \leq L$. This inequality allows all exceptional fixed parts; it does not assume that either ruling divisor is Cartier on Z.

Apply the cotangent-saturation calculation of Lemma 6.3 to the rational maps to $\mathbb{P}^1$ and $\mathbb{P}^{d-1}$. Their kernel foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ have ranks $n-1$ and $n-d+1$ and satisfy

$$c_1(\mathcal{F}_1) \geq (n-1)L - 2R_1, \qquad c_1(\mathcal{F}_2) \geq (n-1)L - dR_2.$$

Writing $\lambda_1 = R_1 \cdot L^{n-1}/L^n$ and $\lambda_2 = R_2 \cdot L^{n-1}/L^n$, tangent semistability gives

$$\frac{n-1-2\lambda_1}{n-1} \leq \frac{n-1}{n}, \qquad \frac{n-1-d\lambda_2}{n-d+1} \leq \frac{n-1}{n}.$$

Rearranging gives the claimed lower bounds. Adding them and using $\lambda_1 + \lambda_2 \leq 1$ proves the final inequality. □

Proposition 6.7. *If $n \geq 8$, the complete linear system $|L|$ is generically finite onto its image. If $n \in \{5, 6, 7\}$ and $|L|$ is not generically finite, its image has dimension $n-1$ and is the degree-three cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$, with vertex $\mathbb{P}^{n-5}$. The corresponding scroll model is*

$$\mathbb{P}_{\mathbb{P}^1}\big(\mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}^{\oplus(n-4)}\big)$$

with its tautological polarization.

Proof. Suppose $m < n$. Proposition 6.4 gives $d = n - m + 2 \geq 3$ and $d \leq m$, or equivalently $2d \leq n + 2$. If $n \geq 8$, the inequality in Lemma 6.6 gives the contradiction.

For $n = 5$, the condition $2d \leq 7$ forces $d = 3$.

For $n = 6$ or 7, the two-ruling forces $d = 3$.

Thus only $n \in \{5, 6, 7\}$ and $d = 3$ remain. Then $m = n - 1$, and Proposition 6.4 gives the stated image and scroll model. □

The remaining images have dimension one less than Z. We next show that their general resolved fibers are rational curves. The ruling of the Segre cone makes the induced rank-one foliation positive along a general complete-intersection curve, so we can apply the following theorem.

Theorem 6.8 ([BM16, Main theorem], See also [KSCT07, Theorem 1]). *Let Z be a normal complex projective variety, let $\Gamma \subset Z_{\mathrm{reg}}$ be a complete curve, and let $\mathcal{F} \subset T_Z$ be a possibly singular foliation which is regular along Γ. If $\mathcal{F}|_\Gamma$ is an ample vector bundle, then the leaf through every point of Γ is algebraic, and the closure of the leaf through a general point of Γ is rationally connected.*

Lemma 6.9. *Assume $m = n-1$, then a smooth connected general fiber of a resolution of $\phi_{|L|}$ is isomorphic to $\mathbb{P}^1$.*

Proof. By Proposition 6.4, $\beta : B \to W$ is an isomorphism. Identify B with W, so $A = A_W$. Let $\rho : S \to W$ be its scroll model, with tautological class $A_S = \rho^* A_W$, and let $R_{1,S} = \varpi^* \mathcal{O}_{\mathbb{P}^1}(1)$ be its ruling class. The projective-bundle formula gives

$$K_S = -(n-1)A_S + R_{1,S}.$$

Apply the construction of Section 6.1 with $B' = S$ and $\tau = \rho$, so $A' = A_S$ and $h' : Y' \to S$. Then $M = (h')^* A_S$, $\overline{M} = (p')_* M$, and $L - \overline{M} \geq 0$. Saturate the image of $(h')^* \Omega^1_S$ in $\Omega^1_{Y'}$. Its determinant is $(h')^* K_S + R$ for an effective divisor R. The induced rank-one kernel foliation, pushed to Z and saturated in T_Z, therefore satisfies

$$c_1(\mathcal{F}) \geq -K_Z + (p')_* (h')^* K_S = (n-1)(L - \overline{M}) + R_1.$$

Here $R_1 = (p')_* (h')^* R_{1,S}$ is the movable divisor of the dominant rational map $Z \dashrightarrow \mathbb{P}^1$ induced by the ruling. In particular, R_1 is effective and nonzero. Since $L - \overline{M} \geq 0$ and L is ample, we obtain $c_1(\mathcal{F}) \cdot L^{n-1} > 0$.

Choose a sufficiently general complete-intersection curve Γ using a large very ample Cartier multiple of L. Normality makes $\mathrm{Sing}(Z)$ have codimension at least two. The singular locus of a saturated foliation on the smooth locus also has codimension at least two. Thus Γ is smooth and avoids both singular loci. The restriction $\mathcal{F}|_\Gamma$ is a line bundle of positive degree, so it is ample. By Theorem 6.8, the closure of the leaf through a general point of Γ is rationally connected.

By generic smoothness, choose a nonempty open subset $U \subset S$ such that $(h')^{-1}(U) \to U$ is smooth. This restriction of h' is a proper smooth family of connected curves, whose genus is constant because U is irreducible. Let $Z^\circ \subset Z_{\mathrm{reg}}$ be a nonempty open subset over which p' is an isomorphism and such that $(p')^{-1}(Z^\circ) \subset (h')^{-1}(U)$. On Z°, the foliation $\mathcal{F}$ is tangent to the fibers of the morphism $h' \circ (p')^{-1} : Z^\circ \to U$.

For a sufficiently general choice of Γ above, take a general point $x \in \Gamma \cap Z^\circ$, and put $s = h'((p')^{-1}(x)) \in U$. The smooth connected fiber $C_s = (h')^{-1}(s) \subset Y'$ maps birationally under p' onto the closure $\Lambda_x \subset Z$ of the leaf through x. The preceding application of Theorem 6.8 shows that Λ_x is rationally connected, so its normalization C_s has genus zero. Constancy of the genus over U now implies that every closed fiber of $(h')^{-1}(U) \to U$ is isomorphic to $\mathbb{P}^1$. Since $\rho : S \to W$ is birational, the general fibers of h' and $f' = \rho \circ h' : Y' \to W$ coincide. Further smooth birational resolutions preserve this conclusion, since their smooth proper generic curve fibers are birational and hence isomorphic. □

6.3. **The residual Segre cases.** Throughout this subsection, we assume in addition that

$$L^n = 3, \qquad 2L \text{ is Cartier}, \qquad m < n.$$

By Proposition 6.7, one has $n \in \{5, 6, 7\}$ and $m = n-1$, and the Stein base is the normal degree-three cone W over $\mathbb{P}^1 \times \mathbb{P}^2$ with vertex $\mathbb{P}^{n-5}$. Identify B with W via β, so $A = A_W$. Keep the smooth model Y' of Section 6.1. The morphism $f' : Y' \to W$ has connected fibers,

and $M = (f')^*A_W$. Taking $r = 2$ in Section 5.1 gives $2F$ Cartier, and the coefficient in Lemma 5.3 becomes

$$\theta = \frac{n-1}{3(n+1)} \sum_{i=0}^{n} a_i.$$

Let C be a smooth connected general fiber of f'. By Lemma 6.9, one has $C \cong \mathbb{P}^1$. Let $\Sigma \subset Z$ be the non-Cartier locus of L.

The volume and Cartierness of $2L$ leave two possible fiber degrees. The codimension of the base locus distinguishes them.

Lemma 6.10. *One has*

$$a = P \cdot C \in \left\{\frac{1}{2}, 1\right\}.$$

Moreover, $\mathcal{B}$ is nonempty and, writing $k = \operatorname{codim}_Z \mathcal{B}$, one has

$$a = \frac{1}{2} \iff k \leq n-1, \qquad a = 1 \iff k = n.$$

Proof. Since $2P = (p')^*(2L)$ is Cartier, we have $2a \in \mathbb{Z}$. The birational morphism $p' : Y' \to Z$ is an isomorphism on a dense open subset met by the general fiber C of $f' : Y' \to W$. Hence $(p')_*C$ is a nonzero effective curve, and ampleness of L gives

$$a = P \cdot C = L \cdot (p')_*C > 0.$$

Since $M \cdot C = 0$, one also has $F \cdot C = a > 0$, so $\mathcal{B}$ is nonempty. Denote by $k = \operatorname{codim}_Z \mathcal{B}$.

Recall that $M = (f')^*A_W$ and $A_W^{n-1} = \deg W = 3$. Since C is a smooth connected general fiber of f', the projection formula yields

$$a_{n-1} = PM^{n-1} = 3(P \cdot C) = 3a.$$

If $k \leq n-1$, by Lemma 5.1 we have

$$3 = a_0 = a_{k-1} > a_k \geq a_{n-1} = 3a.$$

Thus $0 < a < 1$. Since we already know that $2a \in \mathbb{Z}$, it follows that $a = 1/2$. If $k = n$, the same lemma gives $a_{n-1} = a_0 = 3$, so $a = 1$. This completes the proof. □

It remains to exclude the case $P \cdot C = 1/2$, or equivalently, to show that $\mathcal{B}$ is finite. As in the full-dimensional case, we first use the threshold estimate to obtain a lower bound for the codimension of $\mathcal{B}$, and then use this codimension bound to sharpen the threshold estimate.

Lemma 6.11. *Suppose $P \cdot C = 1/2$. Then $\mathcal{B}$ has codimension at least $n-2$, and*

$$\theta \geq \frac{(n-1)^2}{n+1} = n - 3 + \frac{4}{n+1} > n - 3.$$

Proof. Since W has degree three and dimension $n-1$, projection formula gives

$$a_{n-1} = PM^{n-1} = \frac{3}{2}, \qquad a_n = 0.$$

Set $k = \operatorname{codim}_Z \mathcal{B}$; by Lemma 6.10, one has $k \leq n-1$. By Lemma 5.1, one has $a_i \geq 3/2$ for $0 \leq i \leq n-1$ and $a_0 = \cdots = a_{k-1} = 3$. We also have $\theta \leq k$ by Lemma 5.4. Therefore

$$k \geq \theta \geq \frac{n-1}{3(n+1)} \left(3k + \frac{3}{2}(n-k)\right),$$

whence

$$k \geq \frac{n(n-1)}{n+3} = n - 3 + \frac{9-n}{n+3} > n - 3.$$

The last inequality uses $n \in \{5,6,7\}$. Thus $k \geq n-2$, and the first $n-2$ mixed intersections equal three. The two remaining terms through a_{n-1} are at least $3/2$, so

$$\theta \geq \frac{n-1}{3(n+1)}\left(3(n-2)+2\cdot\frac{3}{2}\right) = \frac{(n-1)^2}{n+1}.$$

□

The half-integrality of F is decisive in the next argument: a general fiber C with $P\cdot C = 1/2$ meets exactly one component G of F, with $G\cdot C = 1$. We show that G is also the only p'-exceptional divisor meeting C. Adjunction along C then gives a value for its log discrepancy that contradicts the preceding threshold estimate.

Lemma 6.12. *The possibility $P\cdot C = 1/2$ does not occur.*

Proof. Suppose $P\cdot C = 1/2$. Take C sufficiently general over the smooth locus of W, so that f' is smooth near C. Recall that $C \simeq \mathbb{P}^1$. Write

$$P = M + F, \qquad F = \sum_j b_j D_j,$$

where the D_j are distinct prime divisors and $b_j \in \frac{1}{2}\mathbb{Z}_{>0}$, since $2F$ is Cartier. A general fiber is not contained in any D_j, so each $D_j\cdot C$ is a nonnegative integer. Since $M\cdot C = 0$, one has

$$\frac{1}{2} = F\cdot C = \sum_j b_j (D_j\cdot C).$$

Consequently there is exactly one component G meeting C, and

$$b_G = \frac{1}{2}, \qquad G\cdot C = 1.$$

By Lemma 6.11, the base locus has codimension at least $n-2 \geq 3$. Thus $|L|$ has no fixed divisorial component, and every component of F is p'-exceptional. If a p'-exceptional prime divisor H is not in $\operatorname{Supp} F$, its center is not contained in the base locus. Near the generic point of that center the map $|L|$ is a morphism, so $\dim f'(H) \leq \dim p'(H) \leq n-2$. Such a divisor is vertical over W and misses C. The other components of F also miss C by the displayed intersection sum. Hence G is the only p'-exceptional prime divisor meeting C.

Write the discrepancy formula as

$$K_{Y'} = (p')^* K_Z + \sum_{H \text{ exceptional}} (A_Z(H)-1)H.$$

Since f' is smooth near the general fiber C, its normal bundle is trivial and $K_{Y'}\cdot C = \deg K_C = -2$. Only G contributes to the sum, and $G\cdot C = 1$; hence

$$A_Z(G) - 1 = (K_{Y'} - (p')^* K_Z)\cdot C = -2 + \frac{n-1}{2}.$$

Thus $A_Z(G) = (n-3)/2$. On the other hand, Lemmas 5.3 and 6.11 give

$$A_Z(G) \geq \theta b_G = \frac{\theta}{2} > \frac{n-3}{2},$$

a contradiction. □

With the half-degree case excluded, Lemma 6.10 now gives finiteness of the base locus.

Corollary 6.13. *The geometric base locus $\mathcal{B}$ is finite. In particular, if (X,L) is a K-semistable limit of smooth cubic hypersurfaces, then the image W of $\phi_{|L|}$ has dimension n.*

Proof. Lemmas 6.10 and 6.12 give $\operatorname{codim}_Z \mathcal{B} = n$. Thus $\mathcal{B}$ is finite. For (X, L) a cubic limit, the image of $\phi_{|L|}$ has dimension n by Proposition 6.7 for $n \geq 8$. For $n \in \{5, 6, 7\}$, if the image has dimension $m < n$, then by the finiteness of the base locus and Proposition 4.8, we have L is Cartier and L is very ample by [Fuj90, Corollary 1.7]. This contradicts Proposition 6.7, so the image has dimension n. □

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